Q. $u : [0, L] \times [0, T) \to \mathbb{R}$.

$$u_{tt} = u_{xx}.$$

$$u(x,0) = g, \quad u_t(x,0) = h(x)$$

$$u(0,t) = u(L,t) = 0.$$

How to estimate $|u|$??

A. Use Poincaré inequality and Energy !!

$$E(t) = \frac{1}{2}\int_0^1 u_x^2 + u_t^2 \, dx$$

$$E' = \int u_x u_{xt} + u_t u_{tt}$$

$$= u_x \cancel{u_t}\Big|_0^1 - \int \cancel{u_{xx}} u_t + \int \cancel{u_{tt}} u_t$$

$$= 0$$

$$\Rightarrow E(t) = E(0) = \frac{1}{2}\int g'^2 + h^2.$$

$$\Rightarrow \frac{1}{2}\int_0^1 u_x^2(x,t)\,dx \le E(t)$$

$$= \frac{1}{2}\int_0^1 g'^2 + h^2\,dx$$

$$\Rightarrow \int_0^1 u_x^2(x,t)\,dx \le C_0$$

$a \in [0,1]$

$$u(a,t) = u(a,t) - u(0,t)$$

$$= \int_0^a u_x(x,t)\,dx$$

Hölder $\Rightarrow$

$$\le \left(\int_0^a u_x^2\,dx\right)^{1/2}\left(\int_0^a dx\right)^{1/2}$$

$$\le C_0^{1/2} a^{1/2} \le C_0^{1/2}$$

similarly, $u \ge -C_0^{1/2}$

$$\therefore |u(x,t)| \le \sqrt{\int_0^1 g'^2(x) + h^2(x)\,dx}$$

L^p-norm,

$$\|f\|_{L^p(\Omega)} = \left(\int_\Omega |f|^p \, dx\right)^{1/p}, \quad p > 0.$$

$$\|\nabla f\|_{L^p(\Omega)} = \left(\int_\Omega |\nabla f|^p \, dx\right)^{1/p}$$

where $|\nabla f| = \left(\sum_{i=1}^n \left(\frac{\partial}{\partial x_i} f\right)^2\right)^{1/2}.$

$$\|\nabla^k f\|_{L^p(\Omega)} = \left(\int_\Omega |\nabla^k f|^p \, dx\right)^{1/p}.$$

where $|\nabla^k f| = \left(\sum_{|\alpha|=k} |D^\alpha f|^2\right)^{1/2}.$

· Sobolev inequality

Ω has C^∞ boundary.

$f \in C^\infty(\overline{\Omega})$.

$f = 0$ on $\partial\Omega$

(Ω is bounded).

$$\Rightarrow \|f\|_{L^{\frac{np}{n-p}}(\Omega)} \leq C \|\nabla f\|_{L^p(\Omega)}$$

for $p < n$. $C = C(p, n)$.

$$\sup_\Omega |f| \leq C |\Omega|^{\frac{1}{n} - \frac{1}{p}} \|\nabla f\|_{L^p(\Omega)}$$

for $p > n$. $|\Omega| = \mathrm{vol}(\Omega)$. $C = C(p, n)$.

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proof) [GT] section 7. / Google.

Remark) Given a bounded Ω

$$\|f\|_{L^p(\Omega)} \le C(\Omega, p, q) \|f\|_{L^q(\Omega)}$$

for $q \ge p$.

Pf) (Hw. Hint: Hölder ineq.

Notice. $p < n \Rightarrow \frac{np}{n-p} = \left(\frac{1}{p} - \frac{1}{n}\right)^{-1} > p$.

Remark) Consider $f = 0$ on $\partial\Omega$

$f_k = f + k$, $\quad k \in \mathbb{R}$.

$\Rightarrow \nabla f_k = \nabla f$

$\Rightarrow \|\nabla f_k\|_{L^p} = \|\nabla f\|_{L^p}$.

but $\lim_{k \to \infty} \|f_k\|_{L^p} = +\infty$.

Sobolev - Poincaré ineq

$\partial\Omega$ is C^∞. $f \in C^\infty(\overline{\Omega})$

Ω is bounded

$$\Rightarrow \|f - f_\Omega\|_{L^{\frac{np}{n-p}}} \le C\|\nabla f\|_{L^p}$$

where $f_\Omega = \frac{1}{\mathrm{vol}(\Omega)} \int_\Omega f$

for $p < n$.

$$\boxed{\text{Remark)} \quad \int_\Omega f - f_\Omega = 0.}$$

$$|f - f_\Omega| \le C\|\nabla f\|_{L^p}$$

for $p > n$.

$u_{tt} = \Delta u$ in $\mathbb{R}^2 \times [0, T)$, $u \in C^\infty$

$u(x, 0) = g(x)$, $u_t(x, 0) = h(x)$

where $g, h \in C^\infty(\mathbb{R}^2)$, and

$g(x) = h(x) = 0$ if $|x| \geq R$.

Q. $\sup_{\mathbb{R}^2} |u(x,t)| \leq C(g, h, R)$?

Remark) $E(t) = \frac{1}{2} \int_{\mathbb{R}^2} |Du|^2 + u_t^2 \, dx$

$\Rightarrow \int_{\mathbb{R}^2} |Du|^2 \leq C$

$u(x,t) = 0$ for $|x| \geq |t| + R$.

$C \geq \int_{\mathbb{R}^2} |Du|^2 = \int_{B_{R+|t|}(0)} |Du|^2 = \|Du\|^2_{L^2(B_{R+|t|})}$

$n = 2 = 2 = p \Rightarrow \sup|u| \cancel{\leq} C \|Du\|_{L^2}$

proof) Consider $v = \frac{\partial}{\partial x_1} u$

$\Rightarrow$ $v_{tt} = \Delta v$ in $\mathbb{R}^2 \times [0, T)$

$v(x, 0) = g_1$, $v_t(x, 0) = h_1$

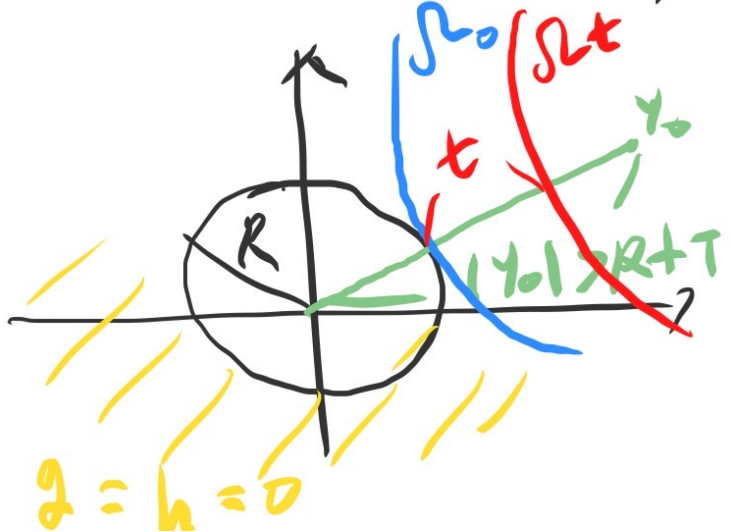
$g_1, h_1 = 0$ in $\mathbb{R}^2 \setminus B_R(0)$.

Step 1) $v(x, t) = 0$ if $|x| > R + |t|$.

$$\tilde{E}(t) = \frac{1}{2} \int_{\Omega_t} |\nabla u|^2 + u_t^2 \, dx$$

$\Omega_t = \{ y \in \mathbb{R}^2 \mid |y - y_0| < |y_0| - R - t \}$

for a $y_0 \in \mathbb{R}^2$ w/ $|y_0| > R + T$.



$\Rightarrow$ $g = h = 0$ in Ω_0.

We know that

$$E' \leq 0 \Rightarrow E(t) \leq E(0) = 0$$

$$\Rightarrow \int_{\Omega_t} |\nabla v|^2 + v_t^2 = 0$$

$$\Rightarrow v = 0 \quad \text{in } \Omega_t.$$

for all $y_0 \in \mathbb{R}^2$ s.t $|y_0| > R+T$

$$\Rightarrow v = 0 \quad \text{in } \mathbb{R}^2 \setminus B_{R+t}(0)$$

for $t \in [0, T)$

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step 2) $\int_{\Omega} |\nabla v|^2 \leq \int_{\mathbb{R}^2} |\nabla g_1|^2 + h_1^2$

where $\Omega = B_{R+T}(0) \subset \mathbb{R}^2$.

for $t \in [0, T)$

By the result of step 1,

$$v = |\nabla v| = |\nabla^2 v| = \cdots = 0$$

in $\mathbb{R}^2 \setminus B_{R+t}(0)$

$\Rightarrow v = |\nabla v| = \cdots = 0$ on $\partial\Omega$

for $t \in [0, T)$

$$E(t) = \frac{1}{2}\int_\Omega |\nabla v|^2 + v_t^2 \, dx$$

$\Rightarrow E' = 0$

$$\Rightarrow \frac{1}{2}\int_\Omega |\nabla v|^2 \leq E(t) = E(0)$$

$$= \frac{1}{2}\int |\nabla g_1|^2 + h_1^2$$

$$\Rightarrow \int_\Omega |\nabla u_1|^2 = \int_\Omega |\nabla v|^2 \leq \int_{\mathbb{R}^2} |\nabla g_1|^2 + h_1^2$$

$$\Rightarrow \int_\Omega |\nabla^2 u(x,t)| \, dx \leq \int_{\mathbb{R}^2} |\nabla^2 g|^2 + |\nabla h|^2$$

for $t \in [0, T)$

step 3) $\|v_1\|_{L^{3/2}(\Omega)} \le C\|v_1\|_{L^2(\Omega)}$ ← Hölder.

$C = C(R+T)$

$\Rightarrow \|\nabla v\|_{L^{3/2}(\Omega)} \le C\|\nabla v\|_{L^2(\Omega)}$

$\|\nabla v\|_{L^2(\Omega)} \le C(\nabla^2 g, \nabla h)$

$\Rightarrow \|\nabla v\|_{L^{3/2}(\Omega)} \le C(R, T, g, h)$

$v = 0$ on $\partial\Omega$

By the sobolev ineq.

$\|v\|_{L^6(\Omega)} \le C\|\nabla v\|_{L^{3/2}(\Omega)}$

$\left(\because p = 3/2,\ n = 2.\ \frac{np}{n-p} = \frac{3}{1/2} = 6\right)$

$\Rightarrow \|u_1\|_{L^6(\Omega)} = \|v\|_{L^6(\Omega)} \le C(R, T, g, h)$

$\Rightarrow \|\nabla u\|_{L^6(\Omega)} \le C(R, T, \nabla^2 g, \nabla h)$

Since $u=0$ on $\partial\Omega$

and $p=6>2=n$.

the Sobolev inequality yields

$$\sup_{\mathbb{R}^2} |u(x,t)| = \sup_{\Omega} |u(x,t)|$$

$$\leq C|\Omega|^{1/3} \|\nabla u\|_{L^6(\Omega)}$$

$$\leq C(R, T, \nabla^2 g, \nabla h)$$

for $t \in [0,T)$ $\blacksquare$

Ex) $\sup_{\mathbb{R}^2} |\nabla u(x,t)| \leq C(R, T, \nabla^3 g, \nabla^2 h)$

proof of Sobolev ineq

$\Omega = [0, L] \times [0, M] \subset \mathbb{R}^2$.

Case 1) $p = 1$.

$$u(x_1, x_2) = \int_0^{x_1} u_1(y_1, x_2)\, dy_1$$

$$\Rightarrow |u(x)| \le \int_0^{x_1} |u_1(y_1, x_2)|\, dy_1$$

$$\le \int_0^{L} |u_1(y_1, x_2)|\, dy_1$$

$$|u(x)| \le \int_0^{M} |u_2(x_1, y_2)|\, dy_2$$

$$\Rightarrow |u(x)|^2 \le \int_0^{L} |u_1(y_1, x_2)|\, dy_1$$

$$\times \int_0^{M} |u_2(x_1, y_2)|\, dy_2$$

$$\|u\|_{L^2}^2 = \int_0^M\int_0^L |u(x)|^2\,dx_1\,dx_2$$

$$\leq \int_0^M\int_0^L \underbrace{\int_0^L |u_1(y_1,x_2)|\,dy_1}_{=f(x_2)} \times \underbrace{\int_0^M |u_2(x_1,y_2)|\,dy_2}_{=g(x_1)}\,dx_1\,dx_2$$

$$= \int_0^M f(x_2) \boxed{\int_0^L g(x_1)\,dx_1}\,dx_2$$

$$= \left(\int_0^M f(x_2)\,dx_2\right)\left(\int_0^L g(x_1)\,dx_1\right)$$

$$= \left(\int_\Omega |u_1(x)|\,dx\right)\left(\int_\Omega |u_2(x)|\,dx\right)$$

$$\leq \left(\int_\Omega |\nabla u(x)|\,dx\right)^2 = \|\nabla u\|_{L^1(\Omega)}^2$$

$$\Omega = [0,L]\times[0,M]$$

Case 2) $1 < p < 2 = n$.

$$\gamma = \frac{(n-1)p}{n-p} = \frac{p}{2-p} > 1$$

$$\| |u|^\gamma \|_{L^2} \le C \| \nabla |u|^\gamma \|_{L^1}$$

$$= C\gamma \int |u|^{\gamma-1} |\nabla u|$$

Holder $\rightarrow$ $$\le C \| |u|^{\gamma-1} \|_{L^{p^*}} \| \nabla u \|_{L^p}$$

where $1 = \frac{1}{p} + \frac{1}{p^*}$, $p^* = \frac{p}{p-1}$.

$$\| |u|^\gamma \|_{L^2} = \left(\int |u|^{2\gamma} \right)^{1/2} = \| u \|_{L^q}^{q/2}$$

$$\| |u|^{\gamma-1} \|_{L^{p^*}} = \left(\int |u|^{\frac{(\gamma-1)p}{p-1}} \right)^{1-\frac{1}{p}} = \| u \|_{L^q}^{q(1-\frac{1}{p})}$$

$$2\gamma = \frac{(\gamma-1)p}{p-1} = \frac{2p}{2-p} = q.$$

$$\Rightarrow \| u \|_{L^q} \le C \| \nabla u \|_{L^p}.$$